WESLEY COLLEGE

By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER ONE 2017
TEST 1: Complex Numbers

AVE

Name: Solutions

Thursday 9th March

Time: 55 minutes **Mark** ~~/50 =~~ 74 %

- Answer all questions neatly in the spaces provided. **Show all working.**
- You are permitted to use the Formula Sheet in **both** sections of the test. S.d.
- You are permitted one A4 page (one side) of notes in the calculator assumed section.

Calculator free section **Suggested time: 20 minutes** **/20**

1. [~~10~~ 11 marks]

Determine each of the following in rectangular form $a+bi$

a) z if $2z-\overline{z}=3-6i$ 3 [~~2~~]

$2a+2bi-a+bi=3-6i$

$\Rightarrow a=3 \ , \ b=-2$ i.e. $z=3-2i$

b) $\dfrac{\overline{3+i}}{(2+i)^2}$ $= \dfrac{3-i}{3+4i} \times \dfrac{3-4i}{3-4i} = \dfrac{9-15i+4i^2}{9-16i^2} = \dfrac{1}{5}-\dfrac{3}{5}i$ [3]

c) one solution to $z^3=8cis\left(\dfrac{3\pi}{4}\right)$ [2]

$z=2cis\frac{\pi}{4}=\sqrt{2}+\sqrt{2}i$

d) $\left(1-\sqrt{3}i\right)^5$ $=\left[2cis\left(-\frac{\pi}{3}\right)\right]^5=32cis\left(-\frac{5\pi}{3}\right)=32cis\frac{\pi}{3}$ [3]

$=16+16\sqrt{3}i$

2. [6 marks]

$(z+2)$ is a factor of $P(z) = z^3 + pz^2 + 14z + 20$.

a) Evaluate p

[2]

$P(-2) = 0 \Rightarrow -8 + 4p - 28 + 20 = 0$

$\Rightarrow p = 4$

b) Rewrite $P(z)$ in the form $P(z) = (z+2)Q(z) + R$

[2]

$P(z) = (z+2)(z^2 + 2z + 10) + 0$

	1	4	14	20
$z=-2$	1	2	10	0

c) Determine all solutions to $P(z) = 0$

[2]

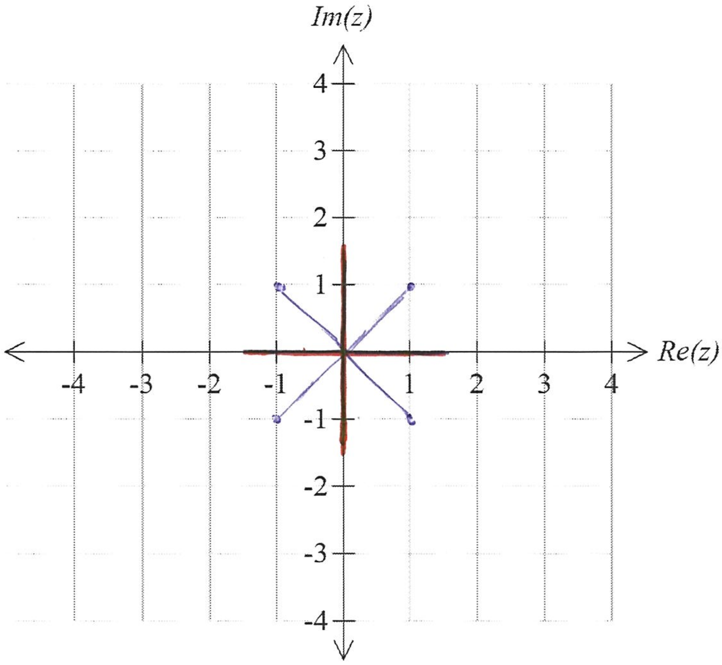
$z = -2$ or $\dfrac{-2 \pm \sqrt{4-40}}{2}$

$= -2, \ -1 \pm 3i$

3. [~~4~~ 3 marks]

When graphed on an Argand diagram, four of the solutions to $z^8 = k$ form a square with vertices $(1, i)$, $(-1, i)$, $(-1, -i)$ and $(1, -i)$.

Evaluate k and then write down the remaining solutions to $z^8 = k$



$k = \left(\sqrt{2}\ \text{cis}\ \frac{\pi}{4}\right)^8 = 16$

Others: $\sqrt{2}$

$\sqrt{2}\,i$

$-\sqrt{2}$

$-\sqrt{2}\,i$

Calculator assumed section **Suggested time: 35 minutes** **/30**

Name: ______________________

4. [4 marks]

$$z = 4\,cis\left(-\frac{\pi}{3}\right) \quad \text{and} \quad \omega = 2\,cis\left(\frac{\pi}{6}\right)$$

For which values of n, $-12 \le n \le 12$, will $\sqrt{z}.\omega^n$ be real?

$$\sqrt{z}.\omega^n = 2\,cis\left(-\frac{\pi}{6}\right) 2^n\,cis\left(\frac{n\pi}{6}\right)$$

$$= 2^{n+1}\,cis\frac{(n-1)\pi}{6}$$

Real when arg $= 0, \pm\pi, \pm 2\pi$ etc

$\Rightarrow n = 1, 7, -5$ or -11 for $-12 \le n \le 12$

5. [4 marks]

Determine, in Cartesian form $a+bi$, all solutions to the equation $z^4 = -16i$

$z^4 = -16i = 16\,cis\left(-\frac{\pi}{2}\right)$

$z_1 = 2\,cis\left(-\frac{\pi}{8}\right)$ $z_2 = 2\,cis\left(-\frac{5\pi}{8}\right)$ $z_3 = 2\,cis\left(\frac{3\pi}{8}\right)$

& $z_4 = 2\,cis\left(\frac{7\pi}{8}\right)$

In Cartesian, with ClassPad

$z_1 = 2\cos\left(-\frac{\pi}{8}\right) + 2i\sin\left(-\frac{\pi}{8}\right) = 1.85 - 0.765i = \sqrt{2+\sqrt{2}} - \sqrt{2-\sqrt{2}}\,i$

$z_2 = 2\cos\left(-\frac{5\pi}{8}\right) + 2i\sin\left(-\frac{5\pi}{8}\right) = -0.765 - 1.85i = -\sqrt{2-\sqrt{2}} - \sqrt{2+\sqrt{2}}\,i$

$z_3 = 2\cos\left(\frac{3\pi}{8}\right) + 2i\sin\left(\frac{3\pi}{8}\right) = 0.765 + 1.85i = \sqrt{2-\sqrt{2}} + \sqrt{2+\sqrt{2}}\,i$

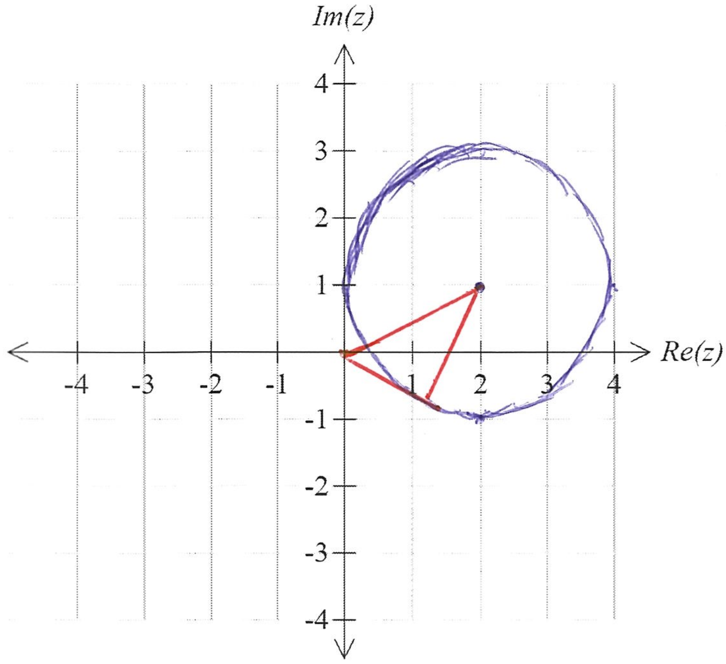
$z_4 = 2\cos\left(\frac{7\pi}{8}\right) + 2i\sin\left(\frac{7\pi}{8}\right) = -1.85 + 0.765i = -\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}}\,i$

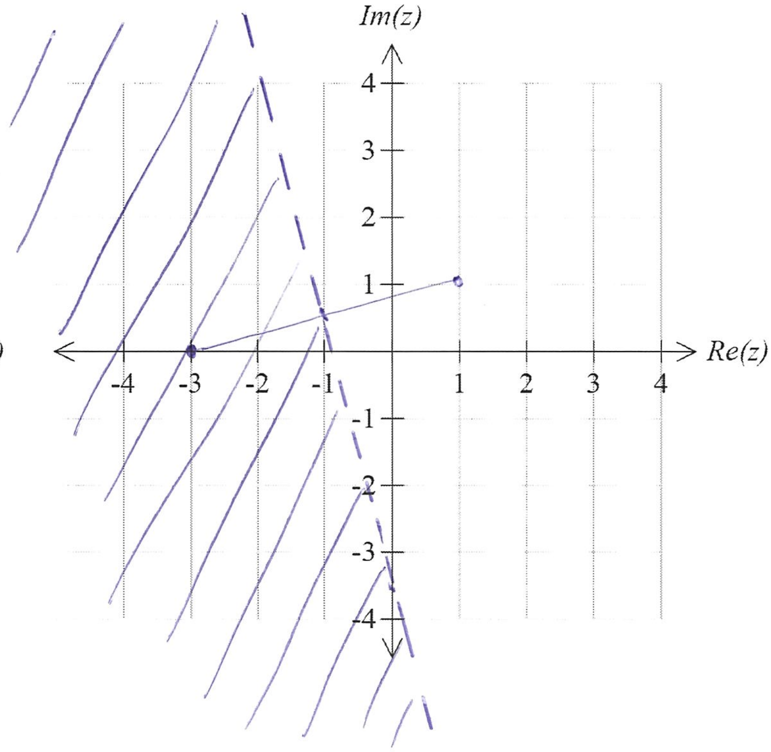
6. [12 marks]

a) On the Argand diagrams given, sketch

(i) $|z-(2+i)|=2$ [2]

(ii) $|z+3|<|z-1-i|$ [4]



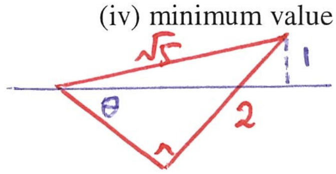


b) For the points defined in (i), determine the:

(iii) maximum value of arg(z) [1]

$\frac{\pi}{2}$

(iv) minimum value of arg(z) [3]

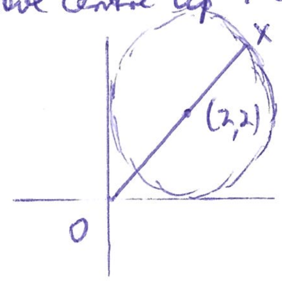


$\theta = \arg(z) = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \tan^{-1}\left(\frac{1}{2}\right) = -0.6435^R$

$(-1.107 + 0.464)$ or $-63.43^\circ + 26.565^\circ = -36.87^\circ$)

(v) maximum value of $|z+i|$ [2]

move centre up 1 unit



max distance of $|z| = OX$

= distance to centre + radius

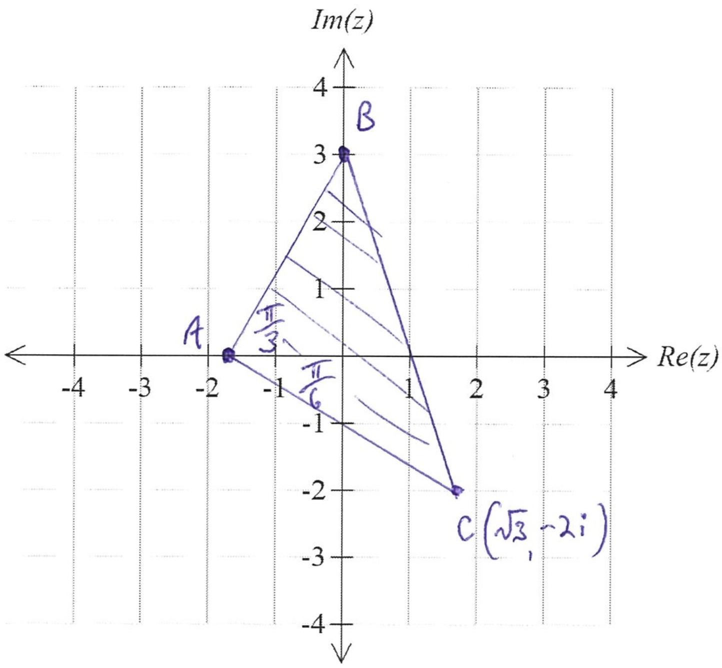
$= 2\sqrt{2} + 2$ units

7. [8 marks]

The line segments joining the points $A\left(-\sqrt{3},0\right)$, $B(0,3i)$ and $C\left(\sqrt{3},-2i\right)$ form a triangle whose interior satisfies two inequalities:

$$\theta_1 \le \arg\left(z+\sqrt{3}\right) \le \theta_2$$

$$\text{and } 5\,\text{Re}(z) + a\,\text{Im}(z) \le b$$



$y = mx + c$

$m = \frac{-5}{\sqrt{3}}$ ✓ $c = 3$

$\therefore 5x + \sqrt{3}y = 3\sqrt{3}$ ✓

Plot ✓

angles ✓

Determine:

a) the values of:

a $\sqrt{3}$ [2]

b $3\sqrt{3}$ ~~[1]~~ 2

θ_1 $-\frac{\pi}{6}$ [2]

θ_2 $\frac{\pi}{3}$ ~~[1]~~ 2

b) the area of triangle ABC [2]

$AB = \sqrt{12} = 2\sqrt{3}$ ✓

$AC = 4$

$\therefore$ area $= 4\sqrt{3}$ units^3 . ✓

(6.928)